Received: *July 10, 2001*, Accepted: *July 24, 2001*

Toy model for tachyon condensation in bosonic string field theory

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Abstract: We study tachyon condensation in a baby version of Witten's open string field theory. For some special values of one of the parameters of the model, we are able to obtain closed form expressions for the stable vacuum state and for the value of the potential at the minimum. We study the convergence rate of the level truncation method and compare our exact results with the numerical results found in the full string field theory.

Keywords: D-branes, Superstring Vacua.

JHEP07(2001)032

Contents

In this letter, we discuss a simple toy model for tachyon condensation in bosonic string field theory. The full string field theory problem [1]–[4] consists of extremising a complicated functional on the Fock space built up from an infinite number of matter and ghost oscillators. As a first simplification, one can consider the variational problem in the restricted Hilbert space of states generated by a single matter oscillator. This problem is still rather non trivial because the restricted Hilbert space still contains an infinite number of states. The model we will consider here is precisely of this form and its behaviour closely resembles the one found in the full theory with level approximation methods. The main simplification lies in the limited number of degrees of freedom and the fact that we don't have to deal with the technicalities of the ghost system.

The motivation for considering such simplified models is twofold. First of all, the level approximation method to the full string theory problem remains largely "experimental": there doesn't seem to be a convincing a priori reason why this approximation scheme converges to the exact answer, nor do we have any information about the rate of convergence except the "experimental" information we have from considering the first few levels. Our toy model will allow for the derivation of exact results on the convergence of the level truncation method albeit in a not fully realistic context.

The second reason for considering toy models is perhaps more fundamental: it would be of considerable interest to obtain the exact solution for the stable vacuum in the full theory. Such an exact solution would allow a detailed description of the physics around the stable vacuum, where interesting phenomena expected to arise [5]. However, despite many efforts, this solution is lacking at the present time.[1] The model we will consider is in some sense the "minimal" problem one should be able to solve if one hopes to find an analytic solution to the full problem.[2]

In section 1 we will give the action of the toy model. In its most general form, the model depends on some parameters that enter in the definition of a star product and are the analogue of the Neumann coefficients in bosonic string field theory. These parameters are further constrained if we insist that the toy model star product satisfies some of the properties that are present in the full string field theory. More specifically, the string field theory star product satisfies the following properties:

- The three-string interaction term is cyclically symmetric.
- The star product is associative.
- Operators of the form $a - a^\dagger$ act as derivations of the star-algebra.

We impose cyclicity of the interaction term in our toy model in section 2. We deduce the equations of motion in section 3. In section 3 we define the star product for the toy model. We discuss the restrictions following from imposing associativity of the star product in section 4. It turns out that we are left with 3 different possibilities, hereafter called case I, II and III. As is the case for the bosonic string field theory we can also look if there is a derivation $D = a - a^\dagger$ of the star-algebra. This further restricts the cases I, II and III to case Id, IId and again Id, respectively. This is explained in section 5, where we also discuss the existence of an identity of the star-algebra.

After having set the stage we can start looking for exact solutions. In section 6 we give the exact results for case I. In particular we are able to write down closed form expressions for the stable vacuum, the effective potential and its branch structure and

[1] See e.g. [6] where a recursive technique was formulated. Other exact solutions are known, see for example [7].

[2] Other toy models for tachyon condensation were considered in [8, 9, 10].

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the convergence rate of the level truncation method. We also compare these results with the behaviour found in bosonic string field theory. In section 7 we mention the other exact solutions we have found. In section 8, we discuss the case IId which perhaps bears the most resemblance to the full string field theory problem. In this case, it is possible to recast the equation of motion in the form of an ordinary second order non-linear differential equation. This equation is not of the Painlevé type and we have not been able to find an exact solution. Here too, it is possible to get very accurate information about the stable vacuum using the level truncation method. We conclude in section 9 with some suggestions for further research.

1. The action

The toy model we consider has the following action (the potential energy is equal to minus the action):

$$S(\psi) = -\frac{1}{2}\langle\psi|(L_0 - 1)|\psi\rangle - \frac{1}{3}\langle V||\psi\rangle|\psi\rangle|\psi\rangle\,, \tag{1.1}$$

where L_0 is the usual kinetic operator $L_0 = a^\dagger a$ and $[a, a^\dagger] = 1$. Let us denote the Fock space which is built up in the usual way by $\mathcal{H}$. The "string field" $|\psi\rangle$ is simply a state in this Fock space $\mathcal{H}$ and can thus be expanded as

$$|\psi\rangle = \psi_0|0\rangle + \psi_1 a^\dagger|0\rangle + \psi_2(a^\dagger)^2|0\rangle + \cdots\,,$$

where the coefficients $\psi_0, \psi_1, \ldots$ are complex numbers. To illustrate the analogy between this toy model and Witten's bosonic string field theory [11], the complex numbers ψ_i in the toy model correspond to the space-time fields in the bosonic string field theory in the Siegel gauge. The term -1 in the kinetic part of the action should be thought of as the zero point energy in the bosonic string. In this way, the state $|0\rangle$ has negative energy.

The interaction term is defined as follows:

$$\langle V||\psi\rangle|\psi\rangle|\psi\rangle = {}_{123}\langle 0|\exp\left(\frac{1}{2}\sum_{i,j=1}^{3} N_{ij}a_i a_j\right)|\psi\rangle_1|\psi\rangle_2|\psi\rangle_3\,. \tag{1.2}$$

The numbers N_{ij} mimic the Neumann coefficient in Witten's string field theory [12]. There they carry additional indices $N_{ij,kl}\,\eta^{\mu\nu}$ where $k, l = 1, \ldots, \infty$ label the different modes of the string and $\mu, \nu = 1, \ldots, 26$ are space-time indices.

We have introduced three copies of the Fock space $\mathcal{H}$. The extra subscript on a state denotes the copy the state is in:

$$\text{if } |\psi\rangle = \sum_m \psi_m a^{\dagger m}|0\rangle \;\in \mathcal{H}\,,$$

$$\text{then } |\psi\rangle_i = \sum_m \psi_m a_i^{\dagger m}|0\rangle_i \in \mathcal{H}_i\,, \qquad \text{for } i = 1, 2, 3\,.$$

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By definition we have the following commutation relations in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$:

$$[a_i, a_j^\dagger] = \delta_{ij} \,. \tag{1.3}$$

Hence the interaction term (1.2) of the toy model is the inner product between the state $|V\rangle = \exp(\frac{1}{2}\sum N_{ij} a_i^\dagger a_j^\dagger)|0\rangle_{123} \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and $|\psi\rangle_1 \otimes |\psi\rangle_2 \otimes |\psi\rangle_3$.

As an example let us calculate the action for $|\psi\rangle = t|0\rangle + u\ a^\dagger|0\rangle$, i.e. the level 1 part of the action (1.1). The kinetic part is obviously

$$-\frac{1}{2}t^2$$

and the interaction is

$$\begin{aligned} &\frac{1}{3}\ {}_{123}\langle 0| \left(1 + \frac{1}{2} N_{ij} a_i a_j\right) \left(t + u\ a_1^\dagger\right) \left(t + u\ a_2^\dagger\right) \left(t + u\ a_3^\dagger\right) |0\rangle_{123} = \\ &\quad = \frac{1}{3}t^3 + \frac{1}{3}\left(N_{12} + N_{13} + N_{23}\right) t u^2 \end{aligned}$$

2. Cyclicity

As is the case for the full string field theory we would like the interaction to be *cyclic*:

$$\langle V||A\rangle|B\rangle|C\rangle = \langle V||B\rangle|C\rangle|A\rangle \,. \tag{2.1}$$

Let us see what restrictions this gives for the matrix N. Imposing the cyclicity (2.1) leads to $N_{11} = N_{22} = N_{33}$, $N_{12} = N_{23} = N_{31}$ and $N_{13} = N_{21} = N_{32}$. Hence N will be of the following form:

$$N = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{13} & N_{11} & N_{12} \\ N_{12} & N_{13} & N_{11} \end{pmatrix} .$$

Because the oscillators a_1, a_2 and a_3 commute among each other, the matrix N can be chosen to be symmetric without losing generality. Hence we have fixed the matrix N to be of the form

$$N_{ij} = \begin{pmatrix} 2\lambda & \mu & \mu \\ \mu & 2\lambda & \mu \\ \mu & \mu & 2\lambda \end{pmatrix} .$$

From this form it is clear that imposing cyclicity in our toy-model forces the star product to be commutative as well.

3. The equation of motion

If we impose the condition that the interaction is cyclically symmetric, the equation of motion reads

$$(a^\dagger a - 1)|\psi\rangle + |\psi\rangle * |\psi\rangle = 0 \,. \tag{3.1}$$

Here we have introduced the star product, it is defined by

$$|\psi\rangle * |\eta\rangle = {}_{23}\langle 0| \exp\left(\frac{1}{2}\sum_{i,j=2}^{3} N_{ij} a_i a_j + \sum_{i=2}^{3} a_1^\dagger N_{1i} a_i + \frac{1}{2} a_1^\dagger N_{11} a_1^\dagger\right) |0\rangle_1 |\psi\rangle_2 |\eta\rangle_3 . \quad (3.2)$$

Let us give some examples of the star product:

$$|0\rangle * |0\rangle = e^{\lambda a^{\dagger 2}} |0\rangle .$$

The star product of two coherent states gives a squeezed state

$$e^{l_1 a^\dagger}|0\rangle * e^{l_2 a^\dagger}|0\rangle = \exp\left(\lambda(l_1^2 + l_2^2) + \mu\ l_1 l_2\right) \exp\left(\lambda a^{\dagger 2} + \mu(l_1 + l_2) a^\dagger\right) |0\rangle .$$

By taking derivatives one can calculate lots of star products e.g.

$$\begin{aligned} |0\rangle * a^\dagger|0\rangle &= \mu a^\dagger e^{\lambda a^{\dagger 2}} |0\rangle , \\ a^\dagger|0\rangle * a^\dagger|0\rangle &= (\mu + \mu^2 a^{\dagger 2}) e^{\lambda a^{\dagger 2}} |0\rangle . \end{aligned}$$

We will now write eqs. (3.1) in terms of the components ψ_n in an expansion

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n (a^\dagger)^n |0\rangle .$$

Let us first take a look at the potential (1.1) in components:

$$V(\psi) = \frac{1}{2}\sum_n n!(n-1)\psi_n^2 + \frac{1}{3}\sum_{m,n,p} m!n!p!\ G_{mnp}\psi_m\psi_n\psi_p , \quad (3.3)$$

where the coefficients G_{mnp} are generated by the function:

$$\begin{aligned} G(z_1, z_2, z_3) &= \exp\left(\frac{1}{2}\sum_{i,j=1}^{3} z_i N_{ij} z_j\right) \\ &\equiv \sum_{mnp} G_{mnp} (z_1)^m (z_2)^n (z_3)^p . \end{aligned}$$

Due to the form of the matrix N, the G_{mnp} are completely symmetric and are zero when the sum $(m+n+p)$ is odd. This last property guarantees that the potential possesses a $\mathbb{Z}_2$ *twist symmetry* just as in the full string field theory. This symmetry acts on the components as $\psi_n \to (-1)^n \psi_n$. As in the full string field theory, the components that are odd under the twist symmetry can be consistently put to zero:

$$\psi_{2n+1} = 0 .$$

Equation (3.1) for the even components becomes

$$(2m-1)\psi_{2m} + \sum_{n,p=0}^{\infty} (2n)!(2p)! G_{2m,2n,2p}\psi_{2n}\psi_{2p} = 0 . \quad (3.4)$$

The trivial solution, $\psi_{2m} = 0$, has $V(\psi) = 0$ and is the one that will correspond to the unstable state. The solution we are looking for will have lower energy and will correspond to a local minimum of the potential.

We can also rewrite the equation of motion (3.1) as a differential equation. Let us use a short hand notation for the string field ψ:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n (a^\dagger)^n |0\rangle \equiv \psi(a^\dagger)|0\rangle \,.$$

If we use $\partial_i = \partial/\partial x_i$, the equation of motion reads

$$\left(x\frac{\partial}{\partial x} - 1\right)\psi(x) + \exp\left(\frac{1}{2}\sum_{i,j=2}^{3} N_{ij}\partial_i\partial_j + x\sum_{i=2}^{3} N_{1i}\partial_i + \frac{1}{2}N_{11}x^2\right) \times \\ \times\, \psi(x_2)\psi(x_3)\big|_{x_2=x_3=0} = 0\,. \tag{3.5}$$

Here we have used that

$$\langle 0|a\ F(a^\dagger)|0\rangle = \frac{\partial}{\partial a^\dagger}F(a^\dagger)\bigg|_{a^\dagger=0} .$$

The resulting equation is a non-linear differential equation of infinite order.

4. Associativity

In the full string field theory the star product is *associative*. We will now check the associativity in our model on a basis of coherent states. The star-product of two coherent states is easy to calculate:

$$e^{l_1 a^\dagger}|0\rangle * e^{l_2 a^\dagger}|0\rangle = A\exp\left(\lambda a^{\dagger 2} + \mu(l_1+l_2)a^\dagger\right)|0\rangle$$

with $A = \exp(\lambda(l_1^2 + l_2^2) + \mu\ l_1 l_2)$. Then using the correlator

$$\langle 0|e^{ka^2+\rho a}e^{la^{\dagger 2}+\sigma a^\dagger}|0\rangle = \frac{1}{\sqrt{1-4kl}}\exp\left(\frac{l\rho^2 + \sigma\rho + k\sigma^2}{1-4kl}\right), \tag{4.1}$$

we find

$$\left(e^{l_1 a^\dagger}|0\rangle * e^{l_2 a^\dagger}|0\rangle\right) * e^{l_3 a^\dagger}|0\rangle = \tag{4.2}$$
$$= AB\frac{1}{\sqrt{1-4\lambda^2}}\exp\left\{\frac{\lambda\mu^2(l_1+l_2)^2 + \mu^2(a^\dagger + l_3)(l_1+l_2) + \lambda(a^\dagger + l_3)^2\mu^2}{1-4\lambda^2}\right\}|0\rangle$$

with $B = \exp(\lambda a^{\dagger 2} + \lambda l_3^2 + \mu a^\dagger l_3)$. Imposing cyclicity among l_1, l_2, l_3 we find that the star-product is associative only in the following three cases, hereafter called case I, II and III:

I. $\mu = 0$, then $N = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$,

II. $2\lambda = \mu - 1$, then $N = \begin{pmatrix} \mu - 1 & \mu & \mu \\ \mu & \mu - 1 & \mu \\ \mu & \mu & \mu - 1 \end{pmatrix}$,

III. $\lambda = 1/2$, then $N = \begin{pmatrix} 1 & \mu & \mu \\ \mu & 1 & \mu \\ \mu & \mu & 1 \end{pmatrix}$.

However, due to the factor $1/\sqrt{1-4\lambda^2}$ in eq. (4.2) the star product of 3 coherent states diverges in the last case. Therefore we should look for another proof of associativity in this case. We will not do this, we just discard this case.

5. Derivation of the star-algebra

Let us now look if $D = a - a^\dagger$ is a derivation of the $*$-algebra:

$$D(A * B) = DA * B + A * DB\,, \qquad \text{where } A \text{ and } B \text{ are two string fields}\,.$$

This is analogous to $\alpha_1^\mu - \alpha_{-1}^\mu$ being a derivation in the full string field theory, see for example [13]. It is easy to see that for D to be a derivation we need

$$\sum_i (a_i - a_i^\dagger)|V\rangle = 0\,. \tag{5.1}$$

Let us calculate the left hand side of (5.1):

$$\sum_i (a_i - a_i^\dagger)|V\rangle = \sum_i \left(\frac{\partial}{\partial a_i^\dagger} - a_i^\dagger\right)|V\rangle = \sum_i (N_{ij} a_j^\dagger - a_i^\dagger)|V\rangle\,.$$

This is zero if and only if $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot (N - 1) = 0$.

- case I

  We need $3(2\lambda - 1) = 0$ so $\lambda = 1/2$. Hence D is a derivation if and only if $N = 1$. We will call this trivial case henceforth case Id.

- case II

  We need $2\mu + \mu - 2 = 0$, hence $\mu = 2/3$. In this case we have

  $$N = \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{pmatrix}. \tag{5.2}$$

  Henceforth, we call this subcase IId.

- case III

  We need $2\mu = 0$ so $\mu = 0$. This reduces to case Id.

For the case Id we will show in section 6 that there is no non-perturbative vacuum. Therefore we consider the value (5.2) as the most important special case that we would like to solve exactly in our toy-model.

In the full string field theory, an important role is played by the *identity string field* I i.e. a string field obeying $I * A = A = A * I$ for (almost) all string fields A.[3] In the toy model, there exists an identity string field I only in case II. In this case we have for the identity I

$$|I\rangle = \frac{\sqrt{2\mu - 1}}{\mu} \exp\left(\frac{1-\mu}{4\mu - 2} a^{\dagger 2}\right) |0\rangle .$$

Using the correlator (4.1), the reader can easily check that

$$|I\rangle * e^{la^\dagger}|0\rangle = e^{la^\dagger}|0\rangle$$

for all coherent states $e^{la^\dagger}|0\rangle$, thus proving that I is the identity. Proving that there is no identity if N does not belong to case II is most easily done by first arguing that the identity should be a gaussian in the creation operator $a^\dagger$ and then showing that one can not find a gaussian which acts as the identity on all coherent states. In case IId the identity string field reduces to

$$|I\rangle = \frac{2}{\sqrt{3}} \exp\left(\frac{1}{2} a^{\dagger 2}\right) |0\rangle . \tag{5.3}$$

6. Exact results in case I

6.1 Closed form expression for the stable vacuum

We now construct the exact solution in the case I, where

$$N = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} .$$

The coefficients $G_{2m,2n,2p}$ entering in the equation of motion (3.4) are particularly simple in this case:

$$G_{2m,2n,2p} = \frac{\lambda^{m+n+p}}{m!n!p!} .$$

Equation (3.4) reduces to

$$\psi_{2m} = \frac{\lambda^m}{(1-2m)m!} g(\lambda)^2 , \tag{6.1}$$

[3] There are some anomalies in the ghost sector, I is not an identity of the star algebra on *all* states, see [13].

where we have defined a function $g(\lambda)$ by

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(2n)!\lambda^n}{n!} \psi_{2n}(\lambda)\,.$$

Multiplying equation (6.1) by $(2m)!\lambda^m/m!$ and summing over m we obtain $g(\lambda)$:

$$g(\lambda) = \left(\sum_n \frac{\lambda^{2n}(2n)!}{(n!)^2(1-2n)} \right)^{-1} = \frac{1}{\sqrt{1-4\lambda^2}}\,.$$

Hence our candidate for the *stable vacuum* $|\text{vac}\rangle$ is

$$|\text{vac}\rangle = \frac{1}{1-4\lambda^2} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(1-2n)} (a^\dagger)^{2n}|0\rangle\,.$$

Using the representation (3.5), it is also possible to derive a generating function for the coefficients ψ_{2n}. Putting $\lambda = -l^2$, the differential equation (3.5) reduces to

$$\left(x\frac{\partial}{\partial x} - 1 \right) \psi(x) + \exp -l^2 \left(\partial_2^2 + \partial_3^2 + x^2 \right) \psi(x_2)\psi(x_3)\Big|_{x_2=x_3=0} = 0\,.$$

This is

$$\left(x\frac{\partial}{\partial x} - 1 \right) \psi(x) + e^{-l^2x^2} c^2 = 0\,,$$

where c is just a number

$$c = e^{-l^2\partial_x^2}\psi(x)\Big|_{x=0}\,.$$

A solution of this differential equation is

$$\psi(x) = \frac{1}{1-4l^4}\phi(lx)\,,$$

where $\phi(x)$ is the function

$$\begin{aligned} \phi(x) &= \exp(-x^2) + \sqrt{\pi}x\ \text{erf}(x) \\ &= -\sum_{m=0}^{+\infty} \frac{(-x^2)^m}{m!\ (2m-1)}\,. \end{aligned}$$

The energy difference between the false and true vacuum can be expressed entirely in terms of the function $g(\lambda)$:

$$V(\text{vac}) = -\frac{1}{6}g(\lambda)^3 = -\frac{1}{6}(1-4\lambda^2)^{-3/2}\,.$$

It is clear that the true vacuum only exists for $|\lambda| < 1/2$ since the value of the potential becomes imaginary outside this range. Also, for $|\lambda| > 1/2$, the state $|\text{vac}\rangle$ is no longer normalisable. Note that for the special case Id, $\lambda = 1/2$, there does not seem to be a true vacuum.

6.2 Closed form expression for the effective potential

We can also determine the exact *effective tachyon potential* $V(t)$ by solving for the $\psi_{2n},\ n>\ 0$ in terms of $t\equiv\psi_0$. The equation for these components becomes:

$$\psi_{2m}(\lambda,t)=\frac{\lambda^m}{(1-2m)m!}(t+h(\lambda,t))^2\,,\qquad \text{for } m>0\,, \tag{6.2}$$

where we have defined

$$h(\lambda,t)=\sum_{n=1}^{\infty}\frac{(2n)!\lambda^n}{n!}\psi_{2n}(\lambda,t)\,.$$

Multiplying eq. (6.2) by $(2m)!\lambda^m/m!$ and summing over m we get a quadratic equation for $h(\lambda,t)$:

$$h(\lambda,t)=(\sqrt{1-4\lambda^2}-1)(t+h(\lambda,t))^2\,.$$

The two solutions $h_\pm$

$$h_\pm=\frac{1}{2(1-\sqrt{1-4\lambda^2})}\left(-2t(1-\sqrt{1-4\lambda^2})-1\pm\sqrt{4t(1-\sqrt{1-4\lambda^2})+1}\right)$$

will give rise to two branches of the effective potential. When we also impose the equation for t, we see that the unstable vacuum $t=0$ and the stable vacuum $t=\frac{1}{1-4\lambda^2}$ lie on the same branch (i.e. the one determined by h_+) just as in the full string field theory. Substituting $h_\pm$ in (6.2) to obtain the coefficients $\psi_{2n\pm}(\lambda,t)$ and substituting those in (3.3) we find the exact form of the two branches of the effective potential $V_\pm(t)$:

$$V_\pm=-\frac{1}{2}t^2+\frac{h_\pm^2}{2(1-\sqrt{1-4\lambda^2})}+\frac{1}{3}(t+h_\pm)^3\,.$$

As is the case in the full bosonic string field theory, the branch $V_+(t)$, which links the unstable and the stable vacuum, terminates at a finite negative value t_*, given in this case by

$$t_*=-\frac{1}{4(1-\sqrt{1-4\lambda^2})}\,. \tag{6.3}$$

At this point, the two branches meet. It is also the only point where they intersect, since $V_->V_+$ for all other values of t.

6.3 The level truncation method

We can also discuss the convergence of the level truncation method in this case. We will focus on the level $(2k,6k)$ approximation to the tachyon potential. This means that we include the fields up to level $2k$ and keep all the terms in the potential involving these fields. In this approximation, the equation for the extremum is just

(3.4) with all sums now running from 0 to k. The solution proceeds just as in the previous section. First one solves for the function $g^{(k)}(\lambda)$:

$$g^{(k)}(\lambda) \equiv \left(\sum_{n=0}^{k} \frac{\lambda^{2n}(2n)!}{(n!)^2(1-2n)} \right)^{-1} = \left(\sqrt{1-4\lambda^2} + E(\lambda,k) \right)^{-1}.$$

The function $E(\lambda, k)$, which represents the error we make by truncating at level $2k$, can be expressed in terms of special functions

$$E(\lambda,k) = \frac{2^{1+2k}\lambda^{2(1+k)}\Gamma(\frac{1}{2}+k)\ {}_2F_1(1,\frac{1}{2}+k,2+k;4\lambda^2)}{\sqrt{\pi}(k+1)!}.$$

The level-truncated expressions for the components of the approximate vacuum state $|\mathrm{vac}^{(k)}\rangle$ and the value of $V_{(2k,6k)}$ at the minimum are given by:

$$\begin{aligned} \psi_{2m}^{(k)} &= \frac{\lambda^m}{(1-2m)m!} g^{(k)}(\lambda)^2 , \\ V_{(2k,6k)}(\mathrm{vac}^{(k)}) &= -\frac{1}{6} g^{(k)}(\lambda)^3 . \end{aligned}$$

The determination of the level-truncated effective tachyon potential also proceeds as before. The result is

$$V_{(2k,6k)\pm}(t) = -\frac{1}{2}t^2 + \frac{h_{\pm}^{(k)\,2}}{2(1-\sqrt{1-4\lambda^2}-E(\lambda,k))} + \frac{1}{3}(t+h_{\pm}^{(k)})^3$$

with

$$\begin{aligned} h_{\pm}^{(k)} = \frac{1}{2(1-\sqrt{1-4\lambda^2}-E(\lambda,k))} & \left(-2t\left(1-\sqrt{1-4\lambda^2}-E(\lambda,k)\right) - 1 \pm \right. \\ & \left. \pm \sqrt{4t(1-\sqrt{1-4\lambda^2}-E(\lambda,k))+1} \right). \end{aligned}$$

Again, the potential has two branches which intersect at a finite negative value $t_*^{(k)}$:

$$t_*^{(k)} = -\frac{1}{4(1-\sqrt{1-4\lambda^2}-E(\lambda,k))} \tag{6.4}$$

A plot of both branches of the potential for $k = 0,\ 1,\ 2$ at $\lambda = 0.4$, as compared to the exact result, is shown in figure 1.

6.4 Convergence properties and comparison to the full string field theory

The results of the previous sections allow us to derive some exact results concerning the convergence properties of the level truncation method in this model and to compare them with the behaviour found in the full string field theory using numerical

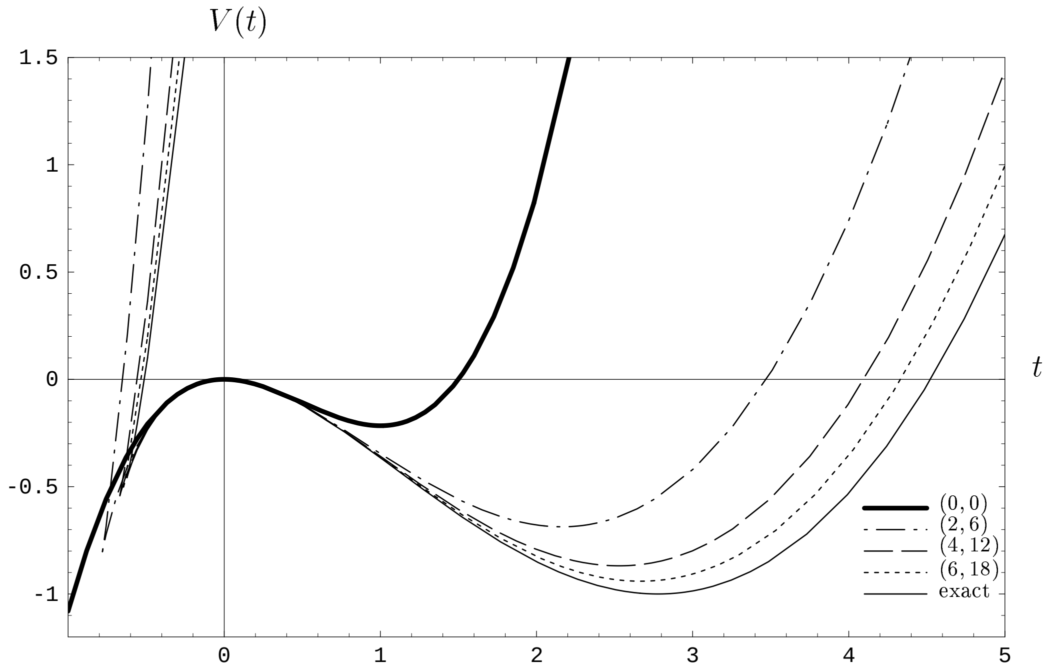

Figure 1: The level-truncated effective potential for $\lambda = 0.4$ at level $(0,0)$, level $(2,6)$, level $(4,12)$ and level $(6,18)$ as compared to the exact result. We have rescaled the potential by a factor $6(1-4\lambda^2)^{3/2}$ so that the minimum occurs at $V = -1$.

methods [4]. For this purpose, we need the asymptotic behaviour of the function $E(\lambda, k)$ for large level k [17]:

$$E(\lambda, k) \sim \frac{2\lambda^2}{\sqrt{\pi}(1-4\lambda^2)} k^{-3/2}(4\lambda^2)^k [1 + \mathcal{O}(k^{-1})]\,, \qquad \text{for } k \to \infty\,. \tag{6.5}$$

Hence the error we make in the level approximation to the coefficients of the true vacuum and the value of the potential at its minimum goes like

$$\begin{aligned} \psi_{2m} - \psi_{2m}^{(k)} &\sim \frac{2^{2k+2}\lambda^{2k+m+2}k^{-3/2}}{\sqrt{\pi}m!(1-2m)(1-4\lambda^2)^{5/2}}\,, \\ V(\mathrm{vac}) - V_{(2k,6k)}(\mathrm{vac}^{(k)}) &\sim -\frac{\lambda^2 k^{-3/2}(4\lambda^2)^k}{\sqrt{\pi}(1-4\lambda^2)^3} \end{aligned}$$

for large level k. We see that, both for the components of the vacuum state and the value of the potential at the minimum, the level truncation method converges to the exact answer in a manner which is essentially exponential as a function of the level: it goes like $k^{-3/2}e^{-k|\ln 4\lambda^2|}$. This exponential behaviour is comparable to the one found "experimentally" in the full string field theory problem in [4]: there, the error was found to behave like $(1/3)^k$.

The effective tachyon potential in the toy model has a finite radius of convergence $|t_*|$ as in the full string field theory. In the level truncation method, the radius of convergence $|t_*^{(k)}|$ rapidly approaches the exact value; indeed, from (6.3), (6.4) and (6.5) we have

$$t_* - t_*^{(k)} \sim \frac{\lambda^2}{8\sqrt{\pi}(1-\sqrt{1-4\lambda^2})^2(1-4\lambda^2)} k^{-3/2}(4\lambda^2)^k \,, \qquad \text{for } k \to \infty \,.$$

In contrast to the string field theory effective potential [1, 4], the toy model effective potential does not display a breakdown of convergence for positive values of t.

7. Other exact solutions

We can also find the exact minimum in case II when $\mu = 1$, i.e. when

$$N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

We need to solve the following equation:

$$\left(x\frac{\partial}{\partial x} - 1 \right) \psi(x) + \exp\left(\partial_2\partial_3 + x(\partial_2 + \partial_3)\right) \psi(x_2)\psi(x_3)\big|_{x_2=x_3=0} = 0 \,.$$

A solution of this equation is $\psi(x) = 1$.

$$|\text{false vac}\rangle = 0|0\rangle \,, \qquad |\text{true vac}\rangle = 1|0\rangle \,.$$

More generally we can also solve

$$N = \begin{pmatrix} 0 & \mu & \mu \\ \mu & 0 & \mu \\ \mu & \mu & 0 \end{pmatrix} ,$$

for general μ, again the solution is $\psi(x) = 1$. However this case is not associative if $\mu \neq 1$.

8. Towards the exact solution in case IId?

8.1 The star product in momentum space

In section 4 we have deduced that $D = a - a^\dagger$ is a derivation of the star algebra. If we write the creation and annihilation operators in terms of the momentum and coordinate operators:

$$\begin{cases} a^\dagger = \dfrac{1}{\sqrt{2}}\,(p + ix), \\ a = \dfrac{1}{\sqrt{2}}\,(p - ix), \end{cases}$$

we see that D is proportional to $\partial/\partial p$. Therefore it is tempting to anticipate that the star product will reduce to an ordinary product in momentum space, and this is indeed the case.[4] If we write the states in momentum representation:

$$|\psi\rangle = \int dp\ \psi(p)|p\rangle_p\,,$$

where the states $|p\rangle_p$ are the eigenstates of the momentum operator $\hat{p}$, normalized in such a way that $\langle p_1|p_2\rangle = \delta(p_1 - p_2)$ — we use the extra subscript to denote which representation we are using. We find

$$|\psi\rangle * |\eta\rangle = \int dp\ \pi^{1/4}\sqrt{\frac{3}{2}}\ \psi(p)\eta(p)|p\rangle_p \tag{8.1}$$

and

$$\langle V||\psi_1\rangle|\psi_2\rangle|\psi_3\rangle = \pi^{1/4}\sqrt{\frac{3}{2}}\int dp\ \psi_1(p)\psi_2(p)\psi_3(p)\,. \tag{8.2}$$

This last equation is easy to prove on a basis of coherent states. If $|\psi_i\rangle = \exp\,(l_i a^\dagger)|0\rangle$, then

$$\langle V||\psi_1\rangle|\psi_2\rangle|\psi_3\rangle = e^{\ l^T N l}\,.$$

Let us verify if we get the same result in momentum space. A coherent state is given by a gaussian in momentum space:

$$e^{la^\dagger}|0\rangle_a = \frac{1}{\pi^{1/4}}\exp\left(-\frac{1}{2}\ l^2 + \sqrt{2}\ l\ k - \frac{k^2}{2}\right).$$

Equation (8.2) then holds by gaussian integration.

As a check on our result we will verify that the state $|I\rangle$ given by (5.3) is the identity in momentum space. In momentum space we have

$$|I\rangle = \sqrt{\frac{2}{3}}\ \frac{1}{\pi^{1/4}}\ 1\,, \qquad \text{as a function in momentum space}\,,$$

therefore we have for arbitrary states ψ

$$|I\rangle * |\psi\rangle = \sqrt{\frac{2}{3}}\ \frac{1}{\pi^{1/4}}\ 1\cdot\pi^{1/4}\sqrt{\frac{3}{2}}\ \psi(k) = \psi(k)\,,$$

as it should be.

[4] In Witten's string field theory the operators $D_n^\mu = \alpha_n^\mu + (-1)^n\alpha_{-n}^\mu$ are derivations of the star algebra. This suggests going to the k — space for the odd matter oscillators α_{2n+1}^μ and to the x — space for the even matter oscillators α_{2n}^μ. See [14] where an analysis along these lines was performed. In Witten's string field theory the star product reduces to a matrix product in the split string formalism [15].

8.2 The equation of motion in momentum space

The equation of motion we want to solve now becomes in momentum space

$$(a^\dagger a - 1)|\psi\rangle + |\psi\rangle * |\psi\rangle = \frac{1}{2}\left(-\frac{\partial^2}{\partial p^2} + (p^2-3)\right)\psi + \pi^{1/4}\sqrt{\frac{3}{2}}\psi(p)^2 = 0\,. \qquad (8.3)$$

If we drop some constants, the differential equation we are left with reads

$$\frac{\partial^2}{\partial p^2}\psi(p) = (p^2-3)\psi(p) + \psi(p)^2\,. \qquad (8.4)$$

So we see that instead of the infinite order differential equation we started with, we have now a second order non-linear differential equation. A large body of literature exists (see e.g. [19]) on second order differential equations that have the Painlevé property, meaning that the solutions to these equations have no movable critical points. Such equations can be transformed to one of 50 equations whose solutions can be expressed in terms of known transcendental functions. Applying the algorithm described in [20], one finds that eq. (8.4) is not of the Painlevé type due to the presence of movable logarithmic singularities. Hence we have been unsuccesful in solving (8.4).

8.3 Numerical results

Even though we are not able to find a closed form solution in this case, we can get good approximate results with the level truncation method. We give the potential including fields up to level 4. It reads:

$$\begin{aligned} V(|\psi\rangle) = {} & \frac{-{\psi_0}^2}{2} + \frac{{\psi_0}^3}{3} - \frac{{\psi_0}^2\,\psi_2}{3} + {\psi_2}^2 + \psi_0\,{\psi_2}^2 + \frac{13\,{\psi_2}^3}{27} + \frac{{\psi_0}^2\,\psi_4}{3} - \frac{34\,\psi_0\,\psi_2\,\psi_4}{9} + \\ & + \frac{41\,{\psi_2}^2\,\psi_4}{27} + 36\,{\psi_4}^2 + \frac{227\,\psi_0\,{\psi_4}^2}{27} + \frac{319\,\psi_2\,{\psi_4}^2}{27} + \frac{1249\,{\psi_4}^3}{81}\,. \end{aligned}$$

We can minimize this action and we find

at level 0: $|\psi\rangle = 1.|0\rangle$
with $V(\psi) = -0.166667$,

at level 2: $|\psi\rangle = (1.05083\ + 0.0870701\ a^{\dagger 2})|0\rangle$
with $V(\psi) = -0.181514$,

at level 4: $|\psi\rangle = (1.0508\ + 0.0867394\ a^{\dagger 2} - 0.000383389\ a^{\dagger 4})|0\rangle$
with $V(\psi) = -0.181521$,

at level 6: $|\psi\rangle = (1.05082\ + 0.0867768\ a^{\dagger 2} - 0.000408059\ a^{\dagger 4} - 0.0000352206\ a^{\dagger 6})|0\rangle$
with $V(\psi) = -0.181523$,

at level 8: $|\psi\rangle = (1.05082\ + 0.0867771\ a^{\dagger 2} - 0.000412528\ a^{\dagger 4} - 0.0000341415\ a^{\dagger 6}$
$+1.788 \cdot 10^{-6}\ a^{\dagger 8})|0\rangle$
with $V(\psi) = -0.181524$,

at level 10: $|\psi\rangle = (1.05082\ + 0.0867771\ a^{\dagger 2} - 0.000412537\ a^{\dagger 4} - 0.0000339848\ a^{\dagger 6}$
$+1.76475 \cdot 10^{-6} a^{\dagger 8} - 4.54233 \cdot 10^{-8} a^{\dagger 10})|0\rangle$
with $V(\psi) = -0.181524$.

We see that the level truncation method clearly converges to some definite answer.

9. Conclusions and topics for further research

We simplified Witten's open string field theory by dropping all the ghosts and keeping only one matter oscillator. The model we constructed closely resembles the full string field theory on the following points:

- There is a false vacuum and a stable vacuum.
- The interaction is given in terms of "Neumann coefficients" and can be written by using an associative star product.
- There is a notion of level truncation which converges rapidly to the correct answer.

For some special values of one of the parameters of the model, we were able to obtain the exact solution for the stable vacuum state and the value of the potential at the minimum.

For other values of the parameters we did not succeed in constructing the exact minimum of the tachyon potential. This does not mean that it is impossible to solve Witten's string field theory exactly. In the full string field theory there is a lot more symmetry around: for example Witten's string field theory has a huge gauge invariance and one could try to solve the equation of motion by making a pure — gauge like ansatz [16].

Therefore maybe a natural thing to do is to set up a toy model that includes some of the ghost oscillators in such a way that there is also a gauge invariance. Another research topic would be to set up a toy model of Berkovits' superstring field theory (see [18] for a recent review). It also should not be too difficult to try to mathematically prove the convergence of the level truncation method in these toy models. This might teach us something about why the level truncation method converges in the full string field theory.

Acknowledgments

This work was supported in part by the European Commission RTN project HPRN-CT-2000-00131. The authors would like to thank Walter Troost for discussions and especially Martin Schnabl for collaboration on several parts of this work. P.J.D.S. is aspirant FWO-Vlaanderen.

References

[1] V.A. Kostelecky and S. Samuel, *On a nonperturbative vacuum for the open bosonic string*, *Nucl. Phys.* **B 336** (1990) 263.

[2] V.A. Kostelecky and R. Potting, *Expectation values, lorentz invariance and cpt in the open bosonic string*, *Phys. Lett.* **B 381** (1996) 89 [`hep-th/9605088`].

[3] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, *J. High Energy Phys.* **03** (2000) 002 [`hep-th/9912249`].

[4] N. Moeller and W. Taylor, *Level truncation and the tachyon in open bosonic string field theory*, *Nucl. Phys.* **B 583** (2000) 105 [`hep-th/0002237`].

[5] A. Sen, *Fundamental strings in open string theory at the tachyonic vacuum*, `hep-th/0010240`;
I. Ellwood and W. Taylor, *Open string field theory without open strings*, *Phys. Lett.* **B 512** (2001) 181 [`hep-th/0103085`];
M. Kleban, A.E. Lawrence and S. Shenker, *Closed strings from nothing*, `hep-th/0012081`;
L. Rastelli, A. Sen and B. Zwiebach, *String field theory around the tachyon vacuum*, `hep-th/0012251`;
A.A. Gerasimov and S.L. Shatashvili, *Stringy Higgs mechanism and the fate of open strings*, *J. High Energy Phys.* **01** (2001) 019 [`hep-th/0011009`].

[6] V.A. Kostelecky and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, *Phys. Rev.* **D 63** (2001) 046007 [`hep-th/0008252`].

[7] T. Takahashi and S. Tanimoto, *Wilson lines and classical solutions in cubic open string field theory*, `hep-th/0107046`.

[8] J.A. Minahan and B. Zwiebach, *Field theory models for tachyon and gauge field string dynamics*, *J. High Energy Phys.* **09** (2000) 029 [`hep-th/0008231`];
J.A. Minahan and B. Zwiebach, *Effective tachyon dynamics in superstring theory*, *J. High Energy Phys.* **03** (2001) 038 [`hep-th/0009246`];
J.A. Minahan and B. Zwiebach, *Gauge fields and fermions in tachyon effective field theories*, *J. High Energy Phys.* **02** (2001) 034 [`hep-th/0011226`].

[9] D. Ghoshal and A. Sen, *Tachyon condensation and brane descent relations in p-adic string theory*, *Nucl. Phys.* **B 584** (2000) 300 [`hep-th/0003278`];

J.A. Minahan, *Mode interactions of the tachyon condensate in p-adic string theory*, *J. High Energy Phys.* **03** (2001) 028 [hep-th/0102071];
J.A. Minahan, *Quantum corrections in p-adic string theory*, hep-th/0105312.

[10] D.J. Gross and V. Periwal, *String field theory, noncommutative Chern-Simons theory and Lie algebra cohomology*, hep-th/0106242.

[11] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[12] D.J. Gross and A. Jevicki, *Operator formulation of interacting string field theory*, *Nucl. Phys.* **B 283** (1987) 1;
Operator formulation of interacting string field theory, 2, *Nucl. Phys.* **B 287** (1987) 225.

[13] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality*, hep-th/0006240.

[14] I. Bars, *Map of Witten's $*$ to Moyal's $*$*, hep-th/0106157.

[15] D.J. Gross and W. Taylor, *Split string field theory, 1*, hep-th/0105059; *Split string field theory, 2*, hep-th/0106036.

[16] M. Schnabl, *String field theory at large B-field and noncommutative geometry*, *J. High Energy Phys.* **11** (2000) 031 [hep-th/0010034].

[17] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover Publications, New York 1970.

[18] N. Berkovits, *Review of open superstring field theory*, hep-th/0105230.

[19] E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen, 1. Gewöhnliche Differentialgleichungen*, Teubner, Stuttgart 1983.

[20] M.J. Ablowitz, A. Ramani and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type, 1*, *J. Math. Phys.* **21** (1980) 715.

JHEP07(2001)032